# ADI Finite Difference Schemes for the Calibration of Stochastic Local Volatility Models: An Adjoint Method

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Joint work with Karel in 't Hout

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# Introduction: Why?

- European call option gives holder the right to buy a given asset at a prescribed date T for a prescribed price K
- $S_\tau$ foreign exchange rate at $\tau \geq 0$
- European call option: payoff $u_0(S_T) = \max(S_T - K, 0)$
- Non-path-dependent European option:
  Fair value at $\tau = 0$ is $e^{-r_d T}\mathbb{E}[u_0(S_T)] = e^{-r_d T}\mathbb{E}[u_0(S_0 e^{X_T})]$
- $r_d, r_f$ risk-free interest rates

# Introduction: Why?

Modelling: $X_\tau = \log(S_\tau / S_0)$

Local volatility (LV) model

$$dX_\tau = (r_d - r_f - \frac{1}{2}\sigma_{LV}^2(X_\tau, \tau))d\tau + \sigma_{LV}(X_\tau, \tau)dW_\tau$$

Stochastic local volatility (SLV) model

$$\begin{aligned} dX_\tau &= (r_d - r_f - \tfrac{1}{2}\sigma_{SLV}^2(X_\tau, \tau)V_\tau)d\tau + \sigma_{SLV}(X_\tau, \tau)\sqrt{V_\tau}dW_\tau^{(1)} \\ dV_\tau &= \kappa(\eta - V_\tau)d\tau + \xi\sqrt{V_\tau}dW_\tau^{(2)} \end{aligned}$$

**Goal: Determine $\sigma_{SLV}$ that reproduces market data**

# Difficulties

- For general $\sigma_{SLV}$ European call values not known exactly
- Calibrate SLV model to underlying LV model

**Gyöngy '86**

LV and SLV model define same European call values if

$$\sigma_{LV}^2(x,\tau) = \sigma_{SLV}^2(x,\tau)\mathbb{E}[V_\tau \| X_\tau = x] = \sigma_{SLV}^2(x,\tau)\frac{\int_0^\infty vp(x,v,\tau)dv}{\int_0^\infty p(x,v,\tau)dv}$$

- $\mathbb{E}[V_\tau \| X_\tau = x]$ dependent on $\sigma_{SLV}$ in a non-trivial way
- For general $\sigma_{LV}$ European call values not known exactly

# Valuation of European options

Common: numerically solving the backward PDE ($t = T - \tau$)

- LV model

$$u_t = \tfrac{1}{2}\sigma_{LV}^2 u_{xx} + (r_d - r_f - \tfrac{1}{2}\sigma_{LV}^2)u_x$$

Maturity $T \to \mathbb{E}[u_0(S_0 e^{X_T})] = u(X_0, T)$

- SLV model

$$\begin{aligned} u_t &= \tfrac{1}{2}\sigma_{SLV}^2 v u_{xx} + \rho\sigma_{SLV} v u_{xv} + \tfrac{1}{2}\xi^2 v u_{vv} \\ &\quad + (r_d - r_f - \tfrac{1}{2}\sigma_{SLV}^2)u_x + \kappa(\eta - v)u_v \end{aligned}$$

Maturity $T \to \mathbb{E}[u_0(S_0 e^{X_T})] = u(X_0, V_0, T)$

# Theoretical alternative: forward PDE

- LV model

$$p_\tau = \frac{\partial^2}{\partial x^2}\left(\frac{1}{2}\sigma_{LV}^2 p\right) - \frac{\partial}{\partial x}\left(\left(r_d - r_f - \frac{1}{2}\sigma_{LV}^2\right)p\right)$$

$$\mathbb{E}[u_0(S_0 e^{X_T})] = \int_{-\infty}^{\infty} u_0(S_0 e^x) p(x, T) dx$$

- SLV model

$$\begin{aligned} p_\tau &= \frac{\partial^2}{\partial x^2}\left(\frac{1}{2}\sigma_{SLV}^2 vp\right) + \frac{\partial^2}{\partial x \partial v}\left(\rho\sigma_{SLV} vp\right) + \frac{\partial^2}{\partial v^2}\left(\frac{1}{2}\xi^2 vp\right) \\ &\quad - \frac{\partial}{\partial x}\left(\left(r_d - r_f - \frac{1}{2}\sigma_{SLV}^2\right)p\right) - \frac{\partial}{\partial v}\left(\kappa(\eta - v)p\right) \end{aligned}$$

$$\mathbb{E}[u_0(S_0 e^{X_T})] = \int_0^{\infty}\int_{-\infty}^{\infty} u_0(S_0 e^x) p(x, v, T) dx dv$$

# Theoretical alternative: forward PDE

- LV model

$$p_\tau = \frac{\partial^2}{\partial x^2}\left(\frac{1}{2}\sigma_{LV}^2 p\right) - \frac{\partial}{\partial x}\left(\left(r_d - r_f - \frac{1}{2}\sigma_{LV}^2\right)p\right)$$

$$\mathbb{E}[u_0(S_0 e^{X_T})] = \int_{-\infty}^{\infty} u(x, T-\tau)p(x,\tau)dx$$

- SLV model

$$\begin{aligned} p_\tau &= \frac{\partial^2}{\partial x^2}\left(\frac{1}{2}\sigma_{SLV}^2 vp\right) + \frac{\partial^2}{\partial x \partial v}\left(\rho\xi\sigma_{SLV} vp\right) + \frac{\partial^2}{\partial v^2}\left(\frac{1}{2}\xi^2 vp\right) \\ &\quad - \frac{\partial}{\partial x}\left(\left(r_d - r_f - \frac{1}{2}\sigma_{SLV}^2 v\right)p\right) - \frac{\partial}{\partial v}\left(\kappa(\eta - v)p\right) \end{aligned}$$

$$\mathbb{E}[u_0(S_0 e^{X_T})] = \int_0^{\infty}\int_{-\infty}^{\infty} u(x, v, T-\tau)p(x,v,\tau)dxdv$$

# Adjoint calibration technique

| | | |
|---|---|---|
| FD Discretization<br>Backward PDE, LV | $\leftrightarrow$ | Adjoint Discretization<br>Forward PDE, LV |
| | | $\updownarrow$ $(\star)$ |
| FD Discretization<br>Backward PDE, SLV | $\leftrightarrow$ | Adjoint Discretization<br>Forward PDE, SLV |

$(\star)$ holds when similar discretizations for the backward PDEs

# Adjoint discretization (LV model)

- FD discretization backward equation ($t = T - \tau$):

$$U'_{LV}(t) = A_{LV}(t)U_{LV}(t), \qquad U_{LV,i}(t) \approx u(x_i, t)$$

- Approximations $P_{LV,i}(\tau) \approx \int_{x_{i-0.5}}^{x_{i+0.5}} p(x,\tau)dx$

$$U_{LV,i_0}(T) = P_{LV}^{\mathrm{T}}(\tau)U_{LV}(T-\tau) = P_{LV}^{\mathrm{T}}(T)U_0 = \sum_i P_{LV,i}(T)U_{0,i}$$

- Adjoint discretization forward equation:

$$P'_{LV}(\tau) = A_{LV}^{\mathrm{T}}(T-\tau)P_{LV}(\tau)$$

# Adjoint discretization (SLV model)

- FD discretization backward equation: $\boldsymbol{U}_{SLV,i,j}(t) \approx u(x_i, v_j, t)$
- Approximations $\boldsymbol{P}_{SLV,i,j}(\tau) \approx \int_{v_{j-0.5}}^{v_{j+0.5}} \int_{x_{i-0.5}}^{x_{i+0.5}} p(x, v, \tau) dx dv$
- Adjoint discretization forward equation:

$$\begin{aligned}
\boldsymbol{U}_{SLV,i_0,j_0}(T) &= \sum_{i,j} \boldsymbol{P}_{SLV,i,j}(\tau) \boldsymbol{U}_{SLV,i,j}(T-\tau) \\
&= \sum_{i} \left( \sum_{j} \boldsymbol{P}_{SLV,i,j}(T) \right) U_{0,i}
\end{aligned}$$

# Adjoint calibration

**If**

- Same discretization in $x$-direction of the backward equations
- Adjoint spatial discretization of the forward equations
- $\sigma^2_{LV}(x_i,\tau) = \sigma^2_{SLV}(x_i,\tau)\frac{\sum_j v_j \boldsymbol{P}_{SLV,i,j}(\tau)}{\sum_j \boldsymbol{P}_{SLV,i,j}(\tau)}$

**Then**

- $P_{LV,i}(\tau) = \sum_j \boldsymbol{P}_{SLV,i,j}(\tau)$
- $U_{LV,i_0}(T) = \boldsymbol{U}_{SLV,i_0,j_0}(T)$

# Adjoint calibration

**If**

- Same discretization in $x$-direction of the backward equations
- Adjoint spatial discretization of the forward equations
- $\sigma^2_{LV}(x_i,\tau) = \sigma^2_{SLV}(x_i,\tau)\frac{\sum_j v_j \boldsymbol{P}_{SLV,i,j}(\tau)}{\sum_j \boldsymbol{P}_{SLV,i,j}(\tau)} \approx \sigma^2_{SLV}(x_i,\tau)\frac{\int_0^\infty vp(x_i,v,\tau)dv}{\int_0^\infty p(x_i,v,\tau)dv}$

**Then**

- $P_{LV,i}(\tau) = \sum_j \boldsymbol{P}_{SLV,i,j}(\tau)$
- $U_{LV,i_0}(T) = \boldsymbol{U}_{SLV,i_0,j_0}(T)$

# Non-linear system of ODEs

- Adjoint discretization yields system of ODEs for $\boldsymbol{P}_{SLV}$
- Corresponding discretization matrix depends on

$$\sigma^2_{SLV}(x_i,\tau) = \sigma^2_{LV}(x_i,\tau)\frac{\sum_j \boldsymbol{P}_{SLV,i,j}(\tau)}{\sum_j v_j \boldsymbol{P}_{SLV,i,j}(\tau)} \tag{1}$$

- Non-linear system of ODEs
- Modified Craig–Sneyd implicit time stepping with inner iteration

# Inner iteration

$\boldsymbol{P}_{SLV,n} = \boldsymbol{P}_{SLV,n-1}$ initial approximation to $\boldsymbol{P}_{SLV}(\tau_n)$;

for $q$ is 1 to $Q$ do

(a) approximate $\sigma^2_{SLV}(x_i, \tau_n)$ by (1);
(b) update $\boldsymbol{P}_{SLV,n}$ by performing a MCS time step;

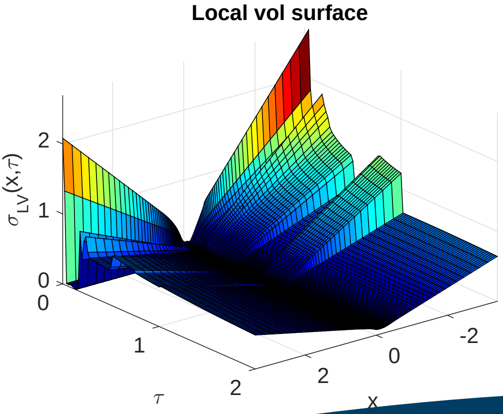
end

- $Q = 2$ performs excellent

# Numerical experiments, EUR/USD rate

$S_0 = 1.0764,\ r_d = 0.03,\ r_f = 0.01$

Local volatility surface up to $\tau = 2$ years

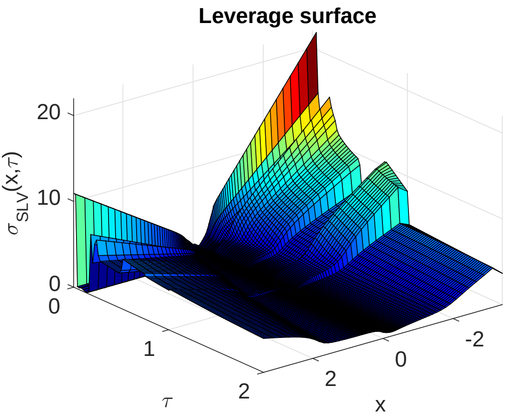


# Stochastic parameters

| | Case 1 | Case 2 |
|---|---|---|
| $\kappa$ | 0.75 | 0.30 |
| $\eta$ | 0.015 | 0.04 |
| $\xi$ | 0.15 | 0.90 |
| $\rho$ | $-0.14$ | $-0.5$ |
| $T$ | $2Y$ | $2Y$ |
| $S_0$ | 1.0764 | 1.0764 |
| $V_0$ | 0.015 | 0.04 |

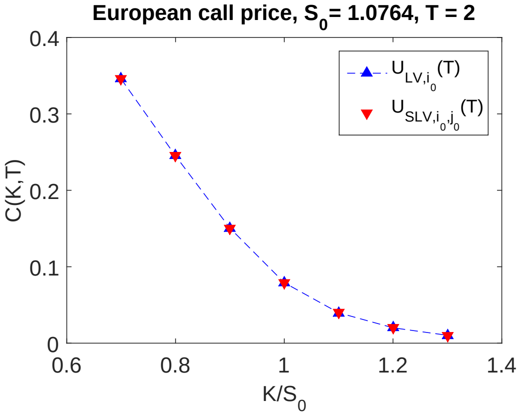
- Case 1: Actual EUR/USD parameters from Clarke (2011)
- Case 2: Challenging parameters from Andersen (2008)

# Leverage surface, Case 1



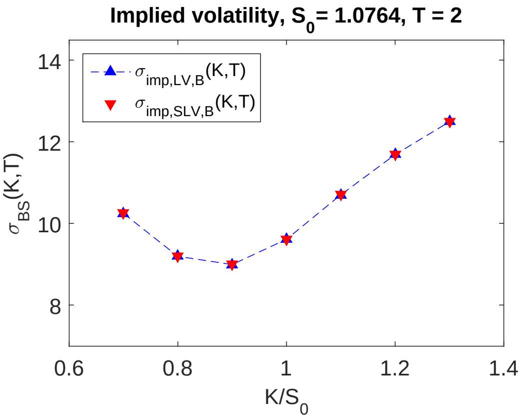
# European call prices, Case 1

Strike $K$, maturity $T \rightarrow U_{0,i} = \max(S_0 e^{x_i} - K, 0)$

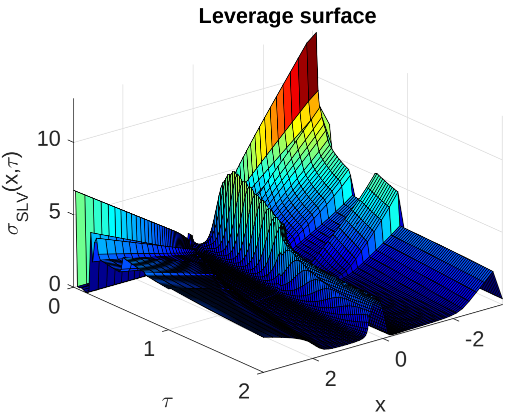


# Implied vols (in %), Case 1

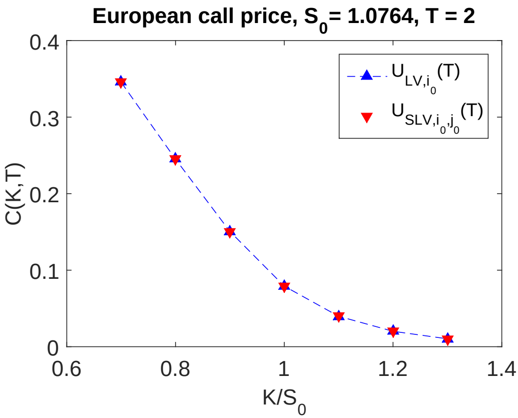
$$U_{LV,i_0}(T) \to \sigma_{imp,LV,B}(K,T), \qquad U_{SLV,i_0,j_0}(T) \to \sigma_{imp,SLV,B}(K,T)$$



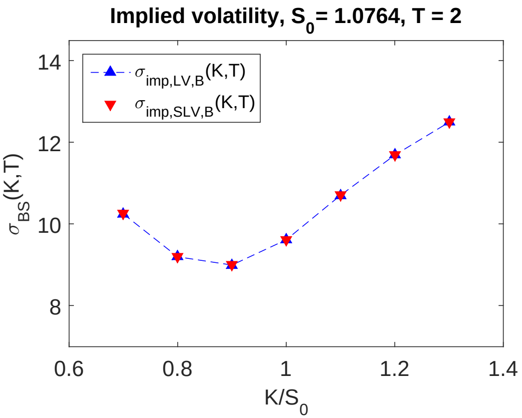
# Leverage surface, Case 2



# European call prices, Case 2



# Implied vols (in %), Case 2



# Numerical results

Absolute implied vol error: $|\sigma_{imp,LV,B}(K,T) - \sigma_{imp,SLV,B}(K,T)|$

$x$-direction: 100 points, $v$-direction: 50 points, $\Delta\tau = 1/100$

| $K/S_0$ | 0.7 | 0.8 | 0.9 | 1.0 | 1.1 | 1.2 | 1.3 |
|---|---|---|---|---|---|---|---|
| $\sigma_{imp,LV,B}$ | 10.23 | 9.19 | 8.99 | 9.61 | 10.70 | 11.68 | 12.49 |
| Case 1 | 4e−3 | 3e−3 | 2e−3 | 1e−3 | 1e−3 | 2e−3 | 2e−3 |
| Case 2 | 2e−3 | 4e−3 | 2e−4 | 5e−3 | 5e−3 | 4e−3 | 3e−4 |

# Calibration time

- Calibration time $\sim$ desired accuracy LV discretization
- Calibration time $\sim$ desired accuracy $v$-direction
- $x$-direction: 100 points, $v$-direction: 50 points, time steps: 100
  $\rightarrow$ Calibration time $\approx 1s$

  (Matlab code, Intel Core i7-3540M 3.00GHz, 8GB RAM)

# Conclusions

- Adjoint calibration for exact match between semidiscrete LV and SLV model
- Time stepping and inner iteration for full discretization
- Fully discrete match upto temporal discretization error
- Spatial error $\gg$ temporal error
- (1) Control discretization error within LV model
  (2) De facto exact calibration of the SLV model

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**Thank You!**

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